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Polynomial extension of some symmetric partition regular structures

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A B S T R A C T

The study of symmetric structures is a new trend in combinatorial number theory. Recently in [9], Di Nasso proved symmetrized versions of some classical theorems like Hindman's theorem, Van der Waerden's theorem and Deuber's theorem. This opens the question of which other classical theorems can be symmetrized, as well as if other symmetric operations allow for such generalizations. Here we give positive answers to both questions, by showing that symmetrization of the polynomial extension of Van der Waerden's and Deuber's Theorem is possible.

1. Introduction

Throughout our article, let $\mathbb{N}$ be the set of all positive integers. For any $r \in \mathbb{N}$, a r-coloring of a set X is a partition of X into r disjoint sets. A core problem in arithmetic Ramsey theory is the characterization of families $\mathcal{F}$ of subsets of a semigroup $(S, \cdot)$ that

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are partition regular, *i.e.* the families have the property that whenever $S = \bigcup_{i=1}^{r} A_i$ is a finite coloring of S, at least one of the A_i is in $\mathcal{F}$. For any r-coloring of S, $A \subset S$ will be called monochromatic if each member of A is in the same color. A cornerstone result in arithmetic Ramsey theory is van der Waerden's theorem [25], which states that for any $l, r \in \mathbb{N}$, and for any r coloring of $\mathbb{N}$, there exists a monochromatic arithmetic progression (AP) of length l. Another important theorem was due to Schur [23] which states that for every finite coloring of $\mathbb{N}$, there exists a monochromatic pattern of the form $\{x, y, x+y\}$. Passing to the map $n \to 2^n$, one can immediately prove the multiplicative version of the Schur theorem (which says $\{x, y, x \cdot y\}$ that is partition regular). Similarly, one can derive the multiplicative version of van der Waerden's theorem, which says that for any $l, r \in \mathbb{N}$, and for any r coloring of $\mathbb{N}$, there exists a monochromatic geometric progression (GP) of length l. In [4], using the methods from Ergodic theory, V. Bergelson proved that for any finite coloring of $\mathbb{Z}$, there exists a monochromatic geo-arithmetic progression (pattern of the form described in Theorem 1.1, can be thought as the combined extension of additive and multiplicative van der Waerden's theorem) of arbitrary length. Later in [3], M. Beiglböck, V. Bergelson, N. Hindman and D. Strauss found an ultrafilter proof.

Theorem 1.1. *[4,3] If $n, r \in \mathbb{N}$, and $\mathbb{Z}$ is r-colored, then there exist $a, b,$ and $d \in \mathbb{N}$ such that the set $\left\{a \cdot (b + i \cdot d)^j : 0 \leq i, j \leq n\right\}$ is monochromatic.*

For any set X, let $\mathcal{P}_f(X)$ be the collection of all nonempty finite subsets of X. Now we recall the notion of IP sets which plays an important role in Ramsey theory.

Definition 1.2 *(IP Sets).* Let $(S, +)$ be a commutative semigroup. A set $A \subseteq S$ is said to be an IP set if there exists an injective sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in S such that

$$A = FS\left(\langle x_n \rangle_{n \in \mathbb{N}}\right) = \left\{\sum_{n \in \alpha} x_n : \alpha \in \mathcal{P}_f(\mathbb{N})\right\}.$$

For $\alpha \in \mathcal{P}_f(\mathbb{N})$, we write $x_\alpha = \sum_{n \in \alpha} x_n$.

The following theorem is another cornerstone theorem in Ramsey theory, known as the Hindman theorem [16].

Theorem 1.3. *For every finite coloring of $\mathbb{Z}$, there exists a monochromatic IP set.*

In [8], Deuber introduced the notion of (m, p, c)-sets and generalized the van der Waerden's theorem.

Definition 1.4 *((m, p, c)-set).* Let $m, p, c \in \mathbb{N}$, and $s = (s_0, \ldots, s_m) \in (\mathbb{Z} \setminus \{0\})^{m+1}$. Then the (m, p, c)-set generated by s is the set

$$D(m,p,c,s)=\left\{\begin{array}{cc} cs_0 & \\ i_0s_0+cs_1, & i_0\in[-p,p] \\ i_0s_0+i_1s_1+cs_2, & i_1,i_0\in[-p,p] \\ \vdots & \vdots \\ i_0s_0+\cdots+i_{m-1}s_{m-1}+cs_m, & i_{m-1},\ldots,i_0\in[-p,p] \end{array}\right\}.$$

The following theorem is due to Deuber [8].

Theorem 1.5. *[8] For any $m,p,c\in\mathbb{Z}$, and for any finite coloring of $\mathbb{Z}$, there exists a monochromatic (m,p,c)-set generated by some $s\in(\mathbb{Z}\setminus\{0\})^{m+1}$.*

Polynomial extensions of classical Ramsey theoretical results are much harder to prove. A pioneering work in this direction is the Polynomial extension of the van der Waerden's theorem, due to V. Bergelson and A. Leibman [17].

Theorem 1.6. *[17] Let $m\in\mathbb{N}$, and $p_1,p_2,\ldots,p_m$ be polynomials with integer coefficients without constant terms. Then for any finite coloring of $\mathbb{N}$, there exist natural numbers a and d such that $\{a+p_i(d):1\le i\le m\}$ is monochromatic.*

In [6], V. Bergelson, J. H. Johnson Jr., and J. Moreira proved the polynomial extension of Theorem 1.5. Studying monochromatic patterns involving both additive and multiplicative structures is relatively new. A pioneering result in this field was due to V. Bergelson and N. Hindman in [5,17], where they independently proved that for any finite coloring of $\mathbb{N}$, there exist monochromatic $a,b,c,$ and d such that $a+b=c\cdot d$. Next in 2017, J. Moreira [5] proved that the family containing patterns of the form $\{a,a+b,a\cdot b\}$ is partition regular. Later in 2019, J.M. Barrett, M. Lupini and J. Moreira [1] proved that the family containing patterns of the form $\{a,a+b,a+b+a\cdot b\}$ is partition regular. For a brief history of additive and multiplicative structures, authors can see the article [20] of J. Moreira.

Recently in [9], by introducing the notions of "symmetric polynomials", M. Di Nasso found many new additive and multiplicative monochromatic patterns.

Definition 1.7 *(Symmetric structure).* For any $n\in\mathbb{N}$, and any $(x_1,x_2,\ldots,x_n)\in\mathbb{N}^n$ a structure $P(x_1,x_2,\ldots,x_n)$ is called symmetric structure if $P(x_1,x_2,\ldots,x_n)=P(x_{i_1},x_{i_2},\ldots,x_{i_n})$ for any $(i_1,i_2,\ldots,i_n)\in S_n$, where S_n is the set of all permutation of $\{1,2,\ldots,n\}$.

If $P(x_1,x_2,\ldots,x_n)$ is a polynomial, then this structure is called symmetric polynomial.

For example, $P(x,y)=x+y+xy$ is a symmetric polynomial. In [9], M. Di Nasso found monochromatic patterns arising from symmetric polynomials. The following theorem is Deuber's theorem for symmetric polynomials.

Theorem 1.8. *[9, Theorem 2.9] Let l and k be two integers where l divides $k-1$, and let $m, L, r \in \mathbb{N}$. Then for every finite coloring $\mathbb{Z} = \cup_{i=1}^{r} C_i$, there exist a color $i \in \{1, 2, \ldots, r\}$, and elements $a_0, \ldots, a_m \in C_i$ such that for every $j = 1, 2, \ldots, m$, and for all $n_0, \ldots, n_{j-1} \in \{0, 1, \ldots, L\}$,*

$$\frac{1}{l}\left((la_j + k)\prod_{s=0}^{j-1}(la_s + k)^{n_s} - k\right) \in C_i.$$

Here we can assume that $(la_j + k) \neq 0, 1, -1$ for all $j \in \{1, 2, \ldots, m\}$.

Inspired by Di Nasso's work, in Section 3, we address the problem of generalizing the polynomial versions of important classical results, particularly polynomial van der Waerden's Theorem 1.6 and polynomial Deuber theorem [6, Theorem 4.9] for symmetric polynomials.

In section 4, we study two new Operations involving both exponential as well as symmetric Patterns.

2. Preliminaries

In this section, we recall some results which are necessary for our work. In section 2.1, we recall the Hales-Jewett theorem and one of its variants. In section 2.2, we recall the algebraic structure of the Stone-Čech compactification of discrete semigroups that we use in section 3.3. In section 2.3, we recall the $\circledast_{l,k}$ operation and some basic facts about it. This operation has been introduced by M. Di Nasso in [9], a very useful operation to find out symmetric Ramsey theoretic patterns.

2.1. Hales-Jewett theorem

Let $\omega = \mathbb{N} \cup \{0\}$. Given a nonempty set $\mathbb{A}$ called alphabet, a finite word is an expression of the form $w = a_1 a_2 \ldots a_n$ with $n \geq 1$, and $a_i \in \mathbb{A}$ for all $i \in \{1, 2, \ldots, n\}$. The quantity n is called the length of w, and denoted by $|w|$. Let v (a variable) be a letter not belonging to $\mathbb{A}$. By a variable word over $\mathbb{A}$, we mean a word w over $\mathbb{A} \cup \{v\}$ that has at least one occurrence of v. For any variable word w, $w(a)$ is the result of replacing each occurrence of v by a.

For any two sets A, B and any function $f : A \to B$, $\mathrm{Dom}(f)$ is the domain of the function f. A located word α is a function from a finite set $\mathrm{Dom}(\alpha) \subseteq \mathbb{N}$ to $\mathbb{A}$. The set of all located words will be denoted by $L(\mathbb{A})$. Note that for located words α, β satisfying $\mathrm{Dom}(\alpha) \cap \mathrm{Dom}(\beta) = \emptyset$, $\alpha \cup \beta$ is also a located word. The following theorem is known as the Hales-Jewett theorem.

Theorem 2.1. *[15, Hales-Jewett Theorem (1963)] For any $t, r \in \mathbb{N}$, there exists a number $HJ(r,t)$ such that, if $N \geq HJ(r,t)$ and $[t]^N$ is r colored then there exists a variable word w such that $\{w(a) : a \in [t]\}$ is monochromatic.*

The word space $[t]^N$ is called Hales-Jewett space or H-J space. The number HJ (r,t) is called the Hales-Jewett number. Let us recall the following variant of the Hales-Jewett theorem due to M. Beiglböck in [2].

Theorem 2.2. *[2, Theorem 3] Let $\mathcal{F}$ be a partition regular family of finite subsets of $\mathbb{N}$ which contains no singletons, and let $\mathbb{A}$ be a finite alphabet. For any finite coloring of $L(\mathbb{A})$ there exist $\alpha \in L(\mathbb{A}), \gamma \in \mathcal{P}_f(\mathbb{N})$ and $F \in \mathcal{F}$ such that $Dom(\alpha), \gamma$ and F are pairwise disjoint and*

$$\{\alpha \cup (\gamma \cup \{t\}) \times \{s\} : s \in \mathbb{A}, t \in F\}$$

is monochromatic.

In [7], using the methods from topological dynamics, V. Bergelson and A. Leibman proved the polynomial extension of the Hales-Jewett Theorem. Then in [26], M. Walter proved it combinatorially. This theorem uses some special notation, which we will state now.

For $q, N \in \mathbb{N}$, $Q = [q]^N$, $\emptyset \neq \gamma \subseteq [N]$ and $1 \leq x \leq q$, $a \oplus x\gamma$ is defined to be the vector b in Q obtained by setting $b_i = x$ if $i \in \gamma$ and $b_i = a_i$ otherwise. In the statement of Theorem [26], we have $a \in Q$ so that $a = \langle \vec{a}_1, \vec{a}_2, \ldots, \vec{a}_d \rangle$ where for $j \in \{1, 2, \ldots d\}$, $\vec{a}_j \in [q]^{N^j}$ and we have $\gamma \subseteq [N] = \{1, 2, \ldots, N\}$. Given $j \in \{1, 2, \ldots, d\}$, let $\vec{a}_j = \langle a_{j,\vec{i}} \rangle_{\vec{i} \in N^j}$. Then $a \oplus x_1\gamma \oplus x_2(\gamma \times \gamma) \oplus \ldots \oplus x_d\gamma^d = b$ where $b = \langle \vec{b}_1, \vec{b}_2, \ldots, \vec{b}_d \rangle$ and for $j \in \{1, 2, \ldots, d\}$, $\vec{b}_j = \langle b_{j,\vec{i}} \rangle_{\vec{i} \in N^i}$ where

$$b_{j,\vec{i}} = \begin{cases} x_j & \text{if } \vec{i} \in \gamma^i \\ a_{j,\vec{i}} & \text{otherwise.} \end{cases}$$

Theorem 2.3. *[26, Polynomial Hales-Jewett Theorem] For any $q, k, d \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that whenever $Q = Q(N) = [q]^N \times [q]^{N \times N} \times \cdots \times [q]^{N^d}$ is k-colored, there exist $a \in Q$ and $\gamma \subseteq [N]$ such that the set of points*

$$\left\{a \oplus x_1\gamma \oplus x_2(\gamma \times \gamma) \oplus \cdots \oplus x_d\gamma^d : 1 \leq x_i \leq q\right\}$$

is monochromatic.

2.2. Preliminaries of the algebra of ultrafilters

For any discrete semigroup $(S, \cdot)$, let βS be the Stone-Čech compactification of S. The operation "$\cdot$" on S naturally extends over βS as: for any $p, q \in \beta S$, $A \in p \cdot q$ if and only if $\{x : x^{-1}A \in q\} \in p$. With this operation "$\cdot$", $(\beta S, \cdot)$ is a compact Hausdorff right topological semigroup. Hence Ellis theorem guarantees that there exist idempotents in $(\beta S, \cdot)$. It can be shown that every member of the idempotents of $(\beta S, \cdot)$ contains an IP

set, that means every idempotent witnesses Hindman's theorem. Using Zorn's lemma one can show that $(\beta S, \cdot)$ contains minimal left ideals (minimal w.r.t. the inclusion). A well known fact is that the union of such minimal left ideals is a minimal two sided ideal, denoted by $K(\beta S, \cdot)$. Here we recall a few well-known classes of sets that are relevant for our work. For details readers can see [19].

Definition 2.4. Let $(S, \cdot)$ be a semigroup, let $n \in \mathbb{N}$ and let $A \subseteq S$. We say that

- A is a thick set if for any finite subset $F \subset S$, there exists an element $x \in S$ such that $Fx = \{fx : f \in F\} \subset A$;
- A is a syndetic set if there exists a finite set $F \subset S$ such that $S = \bigcup_{x \in F} x^{-1}A$, where $x^{-1}A = \{y : xy \in A\}$;
- A is piecewise syndetic if there exists a finite set $F \subset S$ such that $\bigcup_{x \in F} x^{-1}A$ is a thick set. It is well known that A is piecewise syndetic if and only if there exists $p \in K(\beta S, \cdot)$ such that $A \in p$.
- A is central if it belongs to a minimal idempotent in βS.

Minimal idempotents play an important role in Ramsey theory. In [12], Furstenberg proved the Central Sets Theorem, the joint extension of van der Waerden's theorem and Hindman's theorem. In [18] N. Hindman and R. McCutcheon proved the polynomial extension of Central Sets Theorem. In [6], the authors found the polynomial extension of Theorem 1.5. Before we state their result we need the following definition of the polynomial extension of $D(m, p, c)$ sets from [6].

Definition 2.5. [6, Definition 3.1.] Let $m \in \mathbb{N}$, $c : G \to G$ be a homomorphism, and $F = (F_1, \ldots, F_m)$ be an m-tuple, where each F_j is a finite set of functions from G^j to G. For $s = (s_0, \ldots, s_m) \in (G \setminus \{0\})^{m+1}$, the (m, F, c)-set generated by s is the set

$$D(m, F, c, s) = \left\{ \begin{array}{cc} c(s_0) & \\ f(s_0) + c(s_1) & f \in F_1 \\ f(s_0, s_1) + c(s_2) & f \in F_2 \\ \vdots & \vdots \\ f(s_0, \ldots, s_{m-1}) + c(s_m) & f \in F_m \end{array} \right\}.$$

The original theorem is very abstract, but here we mention a special case of [6, Theorem 4.9] which asserts polynomial Deuber's theorem over $\mathbb{Z}$.

Theorem 2.6. *[6, Polynomial Deuber's Theorem] Let $l, k,$ and $m \in \mathbb{N}$, and $c \in \mathbb{Z}$, and $A \subseteq \mathbb{Z}$ be a central set. For $1 \leq i \leq m$, let $F_i \subseteq \mathbb{P}\left(\left(\mathbb{Z}^i, +\right), (\mathbb{Z}, +)\right)$ be a finite collection of polynomials from $(\mathbb{Z}^i, +)$ to $(\mathbb{Z}, +)$ with no constant term. Then there exists an IP-set $\langle S_\alpha \rangle_{\alpha \in \mathcal{P}_f(\mathbb{N})}$ in $\left(\mathbb{Z}^{m+1}, +\right)$ such that $D(m, F, c, s_\alpha) \subseteq A$.*

2.3. A new symmetric operation

After lifting the multiplicative operation over the affine space via an isomorphism, in [9], Di Nasso introduced a new symmetric operation over $\mathbb{Z}$, that we will discuss soon. In this subsection, we describe this operation briefly. First, we need to recall some basic facts from [9].

Definition 2.7 *(Elementary symmetric polynomial).* For $j = 1, 2, \dots, n$, the elementary symmetric polynomial in n variables is the polynomial:

$$e_j (X_1, X_2, \dots, X_n) = \sum_{1 \leq i_1 \leq \dots \leq i_j \leq n} X_{i_1} X_{i_2} \cdots X_{i_j} = \sum_{\emptyset \neq G \subseteq \{1,\dots,n\}} \prod_{s \in G} X_s.$$

For all $a_1, \dots, a_n$, the product $\prod_{j=1}^{n} (a_j + 1) = \sum_{j=1}^{n} e_j (a_1, \dots, a_n) + 1$, and so

$$c = \sum_{j=1}^{n} e_j (a_1, \dots, a_n) \Longleftrightarrow \prod_{j=1}^{n} (a_j + 1) = (c + 1).$$

More generally, for $l, k \neq 0$ it can be easily verified that

$$\prod_{j=1}^{n} (la_j + k) = lc + k \Longleftrightarrow c = \sum_{j=1}^{n} l^{j-1} k^{n-j} e_j (a_1, \dots, a_n) + \frac{k^n - k}{l}.$$

If $k, l \in \mathbb{N}$, then $c \in \mathbb{Z}$ if and only if $l | k (k - 1)$.

The function $\mathfrak{G}_{l,k} (\cdot)$ in the next definition is precisely the same as the value of c in Definition 2.7, which gives a justification for our attention to this function.

Definition 2.8 *((l, k)-symmetric polynomial).* For $l, k \in \mathbb{Z}$ with $l, k \neq 0$ the (l, k)-symmetric polynomial in n variables is

$$\begin{aligned} \mathfrak{G}_{l,k} (X_1, X_2, \dots, X_n) &= \sum_{j=1}^{n} l^{j-1} k^{n-j} e_j (X_1, X_2, \dots, X_n) + \frac{k^n - k}{l} \\ &= \sum_{\emptyset \neq G \subseteq \{1,\dots,n\}} \left(l^{|G|-1} k^{n-|G|} \cdot \prod_{s \in G} X_s \right) + \frac{k^n - k}{l}. \end{aligned}$$

In the following definition, the function $\mathfrak{G}_{l,k} (\cdot)$ is extended to a sequence.

Definition 2.9. Let $\langle x_n \rangle_{n=1}^{\infty}$ be an infinite sequence, and let $l, k \in \mathbb{Z}$ with $l, k \neq 0$. The corresponding (l, k)-symmetric system is the set:

$$\mathfrak{G}_{l,k} (x_n)_{n=1}^{\infty} = \{\mathfrak{G}_{l,k} (x_{n_1}, x_{n_2}, \dots, x_{n_s}) \mid n_1 < n_2 < \dots < n_s\}.$$

Definition 2.10. For $l, k \in \mathbb{Z}$ where $l \neq 0$ divides $k(k-1)$, define

$$a \circledast_{l,k} b = c \Longleftrightarrow (la+k)(lb+k) = (lc+k).$$

So,

$$c = a \circledast_{l,k} b = \frac{1}{l}\left[(la+k)(lb+k) - k\right] = lab + k(a+b) + \frac{k^2-k}{l}.$$

Clearly, $c \in \mathbb{Z}$ if and only if l divides $k^2 - k = k(k-1)$.

In [9, Theorem 2.4], M. Di Nasso proved following symmetrization of the Hindman theorem.

Theorem 2.11 *(Symmetric Hindman theorem). Assume that $l, k \neq 0$ are integers where l divides $k(k-1)$. Then for every finite coloring $\mathbb{Z} = C_1 \cup \ldots \cup C_r$ there exist an injective sequence $\langle x_n \rangle_{n=1}^{\infty}$ of integers, and a color C_i such that $\mathfrak{S}_{l,k}(x_n)_{n=1}^{\infty} \subseteq C_i$.*

More generally, for every injective sequence of integers $\langle x_n \rangle_{n=1}^{\infty}$ and for every finite coloring $\mathfrak{S}_{l,k}(x_n)_{n=1}^{\infty} = C_1 \cup \ldots \cup C_r$ of the corresponding (l,k)-symmetric system, there exist an injective sequence of integers $\langle x_n \rangle_{n=1}^{\infty}$ and a color C_i such that $\mathfrak{S}_{l,k}(y_n)_{n=1}^{\infty} \subseteq C_i$.

Moreover, for positive $l \in \mathbb{N}$, the above partition regularity properties are also true if we replace the integers $\mathbb{Z}$ with the natural numbers $\mathbb{N}$.

Note that for any integers $l \neq 0$,

1. $a \circledast_{l,0} b = lab$;
2. $\mathfrak{S}_{l,0}(a_1, a_2, \ldots, a_n) = a_1 \circledast_{l,0} \cdots \circledast_{l,0} a_n = l^{n-1} a_1 \cdots a_n$.

The iterated $\circledast_{l,k}$-products are exactly the functions defined in Definition 2.8. To verify this fact we recall the following proposition from [9].

Proposition 2.12. *Let $l, k \in \mathbb{Z}$ be such that $l \neq 0$ divides $k(k-1)$. Then for all $a_1, \ldots, a_n \in \mathbb{Z}$,*

$$\begin{aligned} a_1 \circledast_{l,k} \cdots \circledast_{l,k} a_n &= \sum_{\emptyset \neq G \subseteq \{1,\ldots,n\}} \left(l^{|G|-1} k^{n-|G|} \cdot \prod_{s \in G} a_s \right) + \frac{k^n - k}{l} \\ &= \sum_{j=1}^{n} l^{j-1} k^{n-j} e_j(a_1, a_2, \ldots, a_n) + \frac{k^n - k}{l} \\ &= \mathfrak{S}_{l,k}(a_1, a_2, \ldots, a_n). \end{aligned}$$

Fact 2.13. For any $a, b, d \in \mathbb{Z}$, and for $i \in \mathbb{N}$,

$$b \circledast_{l,k} (a + id) = y + iz$$

where $y = l\left(ab + bk + ak\right) + \frac{k^2-k}{l}$ and $z = l^2bd + kld$.

Now, we introduce some new notations which we use in section 3.3.

Definition 2.14. Let $l, k \neq 0$ be integers where l divides $k\left(k-1\right)$. Let $\langle a_n \rangle_{n=1}^{\infty}$ be an injective sequence in $\mathbb{Z}$, $\langle H_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathcal{P}_f\left(\mathbb{N}\right)$, and $B = \{x_n : n \in \mathbb{N}\}$. Then we define the following notations:

1. If $\alpha = \{i_1, i_2, \ldots, i_m\} \in \mathcal{P}_f\left(\mathbb{N}\right)$ with $i_1 < i_2 < \cdots < i_m$, then $a_{\alpha}^{<} = \mathfrak{G}_{l,k}(a_{i_1}, a_{i_2}, \ldots, a_{i_m})$.
2. For $\alpha \in \mathcal{P}_f\left(\mathbb{N}\right)$, $H_\alpha = \bigcup_{i\in\alpha} H_i$.
3. If FS $\langle y_n \rangle_{n=1}^{\infty}$ be a sub-IP set of FS $\langle x_n \rangle_{n=1}^{\infty}$, then for any $\beta = \in \mathcal{P}_f\left(\mathbb{N}\right)$, if $y_\beta = \sum_{j=1}^{p} x_{i_j}$ for some $\{i_j : j = 1, \ldots, p\}$, then define

$$y_{\beta}^{<(B)} = \mathfrak{G}_{l,k}\left(x_{i_1}, x_{i_2}, \ldots, x_{i_p}\right).$$

We call the sequence B a base sequence.

3. Polynomial extension of symmetric configurations

As promised, now we are in the position to provide a proof of symmetric version of Theorem 1.1 using Theorem 2.2. Then we prove the symmetric version of the polynomial van der Waerden's Theorem 1.6 with the help of the polynomial Hales-Jewett Theorem. Later in section 3.3, we prove symmetric version of the polynomial Deuber's theorem.

3.1. Symmetric version of the monochromatic geo-arithmetic progressions

The following theorem is the symmetric version of the Theorem 1.1.

Theorem 3.1. *Let $k, l(\neq 0), r \in \mathbb{Z}$ with $l|k(k-1)$. Then for any r-coloring of $\mathbb{Z}$, and for each $m \in \mathbb{N}$, there exist $x, y, z \in \mathbb{Z}$ such that*

$$\left\{\frac{1}{l}\left[\left(lx+k\right)\left(l\left(y+iz\right)+k\right)^{j}-k\right] : i, j \in \{0, 1, \ldots, m\}\right\}$$

is monochromatic.

Proof. Let $\mathcal{F} = \{\{a, a+d, \ldots, a+kd\} : a, d \in \mathbb{Z}\}$ be the set of all $(k+1)$-terms arithmetic progressions. Clearly, $\mathcal{F}$ is a partition regular family over $\mathbb{Z}$. Choose $\mathbb{A} = \{0, 1, \ldots, m\}$, and define $f : L\left(\mathbb{A}\right) \to \mathbb{N}$ by $f\left(\alpha\right) = \underset{t\in \mathrm{Dom}(\alpha)}{\circledast_{l,k}} t^{(\alpha(t))}$. Color each $\alpha \in L\left(\mathbb{A}\right)$ with the color of $f\left(\alpha\right)$.

Using Theorem 2.2 and the fact 2.13, choose $\alpha, \gamma \in L\left(\mathbb{A}\right)$, and $F \in \mathcal{F}$ such that for all $i, j \in \{0, 1, \ldots, m\}$, we have

$$\begin{aligned}
&f\left(\alpha \cup (\gamma \cup \{a+id\}) \times \{j\}\right)\\
&= \underset{t\in \mathrm{Dom}(\alpha)}{\circledast_{l,k}} t^{(\alpha(t))} \circledast_{l,k} \underset{t\in\gamma}{\circledast_{l,k}} t^{(j)} \circledast_{l,k} (a+id)^{(j)}\\
&= x \circledast_{l,k} b^{(j)} \circledast_{l,k} (a+id)^{(j)}, \text{where } b = \underset{t\in\gamma}{\circledast_{l,k}} t\\
&= x \circledast_{l,k} \left(b \circledast_{l,k} (a+id)\right)^{(j)}\\
&= x \circledast_{l,k} (y+iz)^{(j)}
\end{aligned}$$

is monochromatic for some $y, z \in \mathbb{Z}$.

But $c = x \circledast_{l,k} (y+iz)^{(j)} = \frac{1}{l}\left[(lx+k)\left(l(y+iz)+k\right)^j - k\right]$. This completes the proof □

Suppose $(l,k) = (1,0)$, then the Theorem 3.1 implies Theorem 1.1. The following example gives us a nontrivial application of Theorem 3.1.

Example 3.2. If $(l,k) = (2,1)$, then the pattern

$$\begin{aligned}
&\left\{\frac{1}{2}\left[(2x+1)\left(2(y+iz)+1\right)^j - 1\right] : i,j \in \{0,1,\ldots,m\}\right\}\\
&= \left\{\frac{1}{2}\left[(2x+1)\left((2y+1)+iz\right)^j - 1\right] : i,j \in \{0,1,\ldots,m\}\right\}
\end{aligned}$$

is monochromatic.

3.2. Symmetric version of the polynomial Van der Waerden's theorem

The following theorem is symmetric version of the polynomial van der Waerden's Theorem.

Theorem 3.3. *Let $d,k,l,m \in \mathbb{N}$ where $l|k(k-1)$, and $\left\{a^{(i)} = \left(a_1^{(i)}, a_2^{(i)}, \ldots, a_d^{(i)}\right)\right\}_{i=1}^m \subseteq \mathbb{Z} \setminus \left\{-\frac{k}{l}, -\frac{k+1}{l}\right\}$. Then for any finite coloring of $\mathbb{Z}$ there exist $e, c \in \mathbb{N}$ such that*

$$\left\{\frac{1}{l}\left[(le+k)\prod_{j=1}^{d}\left(la_j^{(i)}+k\right)^{c^j} - k\right]\right\}_{i=1}^m$$

is monochromatic.

Proof. Let $q = \left\{a_1^{(i)}, a_2^{(i)}, \ldots, a_m^{(i)}\right\}_{i=1}^d$, and $N = \mathrm{PHJ}(q,r,d)$. Our r-coloring on $\mathbb{Z} \setminus \left\{-\frac{k}{l}, -\frac{k+1}{l}\right\}$ induces a r-coloring on $[q]^N \times [q]^{N\times N} \times \ldots \times [q]^{N^d}$ via the mapping IM : $a_1 a_2 \ldots a_R \to a_1 \circledast_{l,k} a_2 \circledast_{l,k} \cdots \circledast_{l,k} a_R$. Let $A \subseteq \mathbb{Z}$ be a piecewise syndetic set. So, there exists a finite set E such that $E^{-1}A$ is thick. Note that the set $Q(N)$ is finite. It is a routine exercise to check that the set $\{s : \mathrm{IM}(Q(N)) + s \subseteq E^{-1}A\}$ is infinite. Hence we can translate $IM(Q(N))$ by an element $t \in \mathbb{Z} \setminus \left\{-\frac{k}{l}, -\frac{k+1}{l}\right\}$ such that $\mathrm{IM}(Q(N)) + t \subseteq E^{-1}A$. For $x, y \in Q(N)$, x, y are in same color if and only if $\{\mathrm{Im}(x)+t, \mathrm{Im}(y)+t\} \subset t_1^{-1}A$ for some $t_1 \in E$. Hence by Theorem 2.3, we have a monochromatic combinatorial line of the form

$$\left\{a \oplus x_1\gamma \oplus x_2(\gamma \times \gamma) \oplus \ldots \oplus x_d\gamma^d : 1 \le x_i \le q;\ 1 \le i \le d\right\}.$$

Now, each $a \oplus x_1\gamma \oplus x_2(\gamma \times \gamma) \oplus \ldots \oplus x_d\gamma^d$ is mapped to

$$t + t_1 \circledast_{l,k} b_1 \circledast_{l,k} b_2 \circledast_{l,k} \ldots \circledast_{l,k} b_s \circledast_{l,k} x_1^{(|\gamma|)} \circledast_{l,k} x_2^{(|\gamma|^2)} \circledast_{l,k} \ldots \circledast_{l,k} x_d^{\left(|\gamma|^d\right)},$$

where $x_i \in [q]$, for all $i \in \{1, 2, \ldots, d\}$.

Let $d = t + t_1 \circledast_{l,k} t \circledast_{l,k} b_1 \circledast_{l,k} b_2 \circledast_{l,k} \ldots \circledast_{l,k} b_s$, and therefore the other patterns are of the form $d \circledast_{l,k} x_1^{(|\gamma|)} \circledast_{l,k} x_2^{(|\gamma|^2)} \circledast_{l,k} \ldots \circledast_{l,k} x_d^{\left(|\gamma|^d\right)}$ where $x_i \in [q]$, for all $i \in \{1, 2, \ldots, d\}$.

Now,

$$\begin{aligned} & d \circledast_{l,k} x_1^{(|\gamma|)} \circledast_{l,k} x_2^{(|\gamma|^2)} \circledast_{l,k} \ldots \circledast_{l,k} x_d^{\left(|\gamma|^d\right)} \\ &= \frac{1}{l}\left((le+k)(lx_1+k)^c (lx_2+k)^{c^2} \ldots (lx_d+k)^{c^d} - k\right), \end{aligned}$$

where $x_i \in [q]$, for all $i \in \{1, 2, \ldots, d\}$.

This proves the theorem. □

The following application of Theorem 3.3 is nontrivial.

Example 3.4. Let $(l, k) = (3, 1)$ and $n \in \mathbb{N}$. Let the finite sequence $\left\langle a^{(i)}\right\rangle_{i=1}^n$ be defined by: for $i \in \{1, 2, \ldots, n\}$, $a^{(i)} = (0, 0, \ldots, 1, \ldots, 0)$, where 1 is at the i^{th} coordinate.

Then by Theorem 3.3, there exist $x, y \in \mathbb{N}$ such that

$$\left\{\frac{1}{3}\left((3x+1)4^y - 1\right), \frac{1}{3}\left((3x+1)4^{y^2} - 1\right), \ldots, \frac{1}{3}\left((3x+1)4^{y^n} - 1\right)\right\}$$

is monochromatic.

3.3. Symmetric version of the polynomial Deuber's theorem

In this section we study the polynomial maps from $(\mathbb{Z}, +)$ to $(\mathbb{Z}, \circledast_{l,k})$. As both $(\mathbb{Z}, +)$ and $(\mathbb{Z}, \circledast_{l,k})$ are abelian groups, we can define group polynomials from $(\mathbb{Z}, +)$ to $(\mathbb{Z}, \circledast_{l,k})$. Let us recall the definition of group polynomials.

Definition 3.5 *(Polynomial map).* [6] Given a map $f : H \to G$ between countable commutative groups, we say that f is a polynomial map of degree 0 if it is constant. We say that f is a polynomial map of degree d, where $d \in \mathbb{N}$, if it is not a polynomial map of degree $d-1$ and for every $h \in H$, the map $x \mapsto f(x+h) - f(x)$ is a polynomial of degree $\leq d-1$. Finally we denote by $\mathbb{P}(G, H)$ the set of all polynomial maps $f : H \to G$ with $f(0) = 0$.

One can easily check that a map from $(\mathbb{Z}^m, +)$ to $(\mathbb{Z}, \circledast_{l,k})$ with m variables, and degree n given by

$$P(x_1, x_2, \ldots, x_m) = \underset{i_1+i_2+\cdots+i_m \leq n}{\circledast_{l,k}} \alpha_{i_1 i_2 \cdots i_m}^{\left(x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m}\right)}.$$

is a polynomial map. As $(\mathbb{Z}, +)$ and $(\mathbb{Z}, \circledast_{l,k})$ are two different groups, we cannot apply [6, Theorem 4.9.] directly to conclude polynomial Deuber's Theorem for polynomials from $(\mathbb{Z}, +)$ to $(\mathbb{Z}, \circledast_{l,k})$. So we need a different version of the polynomial Deuber's Theorem. Before that, let us define the notion of symmetric $D(m, p, c)$-set.

Definition 3.6. (Symmetric $D(m, p, c)$-set) Let $m \in \mathbb{N}$, and $\vec{F}$ be an m-tuple $\{F_1, F_2, \ldots, F_m\}$, where $F_i \subseteq \mathbb{P}\left(\left(\mathbb{Z}^i, +\right), \left(\mathbb{Z}, \circledast_{l,k}\right)\right)$ for each $i \in \{1, 2, \ldots, m\}$. Then for any given IP-set $\langle S_\alpha \rangle_{\alpha \in \mathcal{P}_f(\mathbb{N})} \subseteq (\mathbb{Z} \setminus \{0\})^{m+1}$ and a collection of $m+1$ base sequences $B = \{B_0, B_1, \ldots, B_m\}$, define

$$D\left(m, \vec{F}, \circledast_{l,k}, S_\alpha^<, B\right) = \left\{ \begin{array}{lr} \mathfrak{G}_{l,k}\left(S_{\alpha,0}^{<(B_0)}\right) & \\ \mathfrak{G}_{l,k}\left(S_{\alpha,1}^{<(B_1)}, f(S_{\alpha,0})\right) & f \in F_1 \\ \mathfrak{G}_{l,k}\left(S_{\alpha,2}^{<(B_2)}, f(S_{\alpha,0}, S_{\alpha,1})\right) & f \in F_2 \\ \vdots & \vdots \\ \mathfrak{G}_{l,k}\left(S_{\alpha,m}^{<(B_m)}, f(S_{\alpha,0}, \ldots, S_{\alpha,m-1})\right) & f \in F_m \end{array} \right\}.$$

We need the following corollary of [6, Theorem 4.10.].

Corollary 3.7. *Let $\langle y_\alpha \rangle_{\alpha \in \mathcal{F}}$ be an IP-set in $(\mathbb{Z}, +)$, $F \subseteq \mathbb{P}((\mathbb{Z}, +), (\mathbb{Z}, \circledast_{l,k}))$ be finite, and $A \subseteq (\mathbb{Z}, \circledast_{l,k})$ be central. Then there exist a sequence $\langle x_n \rangle_{n=1}^\infty$ in $(\mathbb{Z}, \circledast_{l,k})$ and a sub-IP set $\langle z_\beta \rangle_{\beta \in \mathcal{F}}$ of $\langle y_\alpha \rangle_{\alpha \in \mathcal{F}}$ such that for all $f \in F$, and for all $\beta \in \mathcal{F}$ we have*

$$\mathfrak{G}_{l,k}\left(x_\beta^<, f(z_\beta)\right) \in A.$$

Proof. By Theorem [6, Theorem 4.10.], there exists such a sequence and for all $\beta \in \mathcal{F}$,

$$x_\beta \circledast_{l,k} f(z_\beta) = \mathfrak{G}_{l,k}\left(x_\beta^<, f(z_\beta)\right) \in A. \quad \square$$

Now we are in position to prove the symmetric version of Theorem 2.6.

Theorem 3.8. *Let $l, k, m \in \mathbb{Z}$, and $\mathbb{Z} = \cup_{i=1}^{r} C_i$ be any finite coloring of $\mathbb{Z}$. For $1 \leq i \leq m$, let $F_i \subseteq \mathbb{P}\left(\left(\mathbb{Z}^i, +\right), \left(\mathbb{Z}, \circledast_{l,k}\right)\right)$ be a finite collection of polynomials. Then there exist $j \in \{1, 2, \ldots, r\}$, an IP-set $\langle S_\alpha \rangle_{\alpha \in \mathcal{P}_f(\mathbb{N})}$ in $\left(\mathbb{Z}^{m+1}, +\right)$ and a collection of $m+1$ base sequences $B = \{B_0, B_1, \ldots, B_m\}$ such that for all $\alpha \in \mathcal{P}_f(\mathbb{N})$*

$$D\left(m, \vec{F}, \circledast_{l,k}, S_\alpha^{<}, B\right) \subset C_j.$$

Remark 3.9. *In Theorem 3.8 if each F_i is identity polynomial, then for all $j \in \{1, 2, \ldots, m\}$, we have $\mathfrak{G}_{l,k}\left(S_{\alpha,j}^{<(B_j)}\right) \in C_i$.*

Proof. (Proof of Theorem 3.8:) To avoid complicated calculations, we will only show the first two steps. One can complete the rest of the proof similarly. Suppose that C_i is central for some $i \in \{1, 2, \ldots, r\}$. Hence there exists an injective sequence $\langle x_{n,0} \rangle_{n=1}^{\infty}$ such that $\mathfrak{G}_{l,k}\left(x_{n,0}\right)_{n=1}^{\infty} \subset C_i$. Passing to a subsequence, we can assume that the finite sums are distinct; i.e. $x_{\alpha,0} \neq x_{\beta,0}$ if $\alpha \neq \beta$.

Let $B_0 = \{x_{n,0} : n \in \mathbb{N}\} \subseteq C_i$. From Corollary 3.7, we have a sub-IP-set $\mathrm{FS}\left(\langle y_{n,0} \rangle_{n=1}^{\infty}\right)$ of $\mathrm{FS}\left(\langle x_{n,0} \rangle_{n=1}^{\infty}\right)$ and an injective sequence $B_1 = \{x_{n,1} : n \in \mathbb{N}\}$ such that for all $f \in F_1$, and $\beta \in \mathcal{P}_f(\mathbb{N})$, $\mathfrak{G}_{l,k}\left(x_{\beta,1}^{<(B_1)}, f\left(y_{\beta,0}\right)\right) \in C_i$.

Again passing to a subsequence if necessary, assume that $x_{\alpha,1} \neq x_{\beta,1}$ if $\alpha \neq \beta$. So, we have an IP-set $S_\alpha = (S_{\alpha,0}, S_{\alpha,1})$, and two base sequences B_0, B_1 such that for all $f \in F_1$, and $\alpha \in \mathcal{P}_f(\mathbb{N})$ we have

$$\mathfrak{G}_{l,k}\left(S_{\alpha,0}^{<(B_0)}\right) \in C_i \text{ and}$$
$$\mathfrak{G}_{l,k}\left(S_{\alpha,1}^{<(B_1)}, f\left(S_{\alpha,0}\right)\right) \in C_i.$$

Let $S_n = (S_{n,0}, S_{n,1}) = (y_{n,0}, x_{n,1})$. From Corollary 3.7, there exist a sub-IP-set $\mathrm{FS}\left(\langle S_n' \rangle_{n=1}^{\infty}\right)$ of S_α and an injective sequence $B_2 = \{x_{n,2} : n \in \mathbb{N}\}$ such that for all $f \in F_2$ and $\beta \in \mathcal{P}_f(\mathbb{N})$, $\mathfrak{G}_{l,k}\left(x_{\beta,2}^{<(B_2)}, f\left(S_\beta'\right)\right) \in C_i$. Again passing to a subsequence if necessary we may assume $x_{\alpha,2} \neq x_{\beta,2}$ if $\alpha \neq \beta$. So, we have an IP-set $S_\alpha = (S_{\alpha,0}, S_{\alpha,1}, S_{\alpha,2})$, and three base sequences B_0, B_1, B_2 such that for all $f \in F_2$, and $\alpha \in \mathcal{P}_f(\mathbb{N})$ we have

$$\mathfrak{G}_{l,k}\left(S_{\alpha,0}^{<(B_0)}\right) \in C_i,$$
$$\mathfrak{G}_{l,k}\left(S_{\alpha,1}^{<(B_1)}, f\left(S_{\alpha,0}\right)\right) \in C_i \text{ and}$$
$$\mathfrak{G}_{l,k}\left(S_{\alpha,2}^{<(B_2)}, f\left(S_{\alpha,0}, S_{\alpha,1}\right)\right) \in C_i.$$

Here, $S_\alpha = (S_{\alpha,0}, S_{\alpha,1}, S_{\alpha,2}) = \left(S_{\alpha,0}', S_{\alpha,1}', b_{\alpha,2}\right)$. Iterating this argument we have the desired result. □

The following examples of partition regular structures are new.

Example 3.10. Let $m, n \in \mathbb{N}$, $(l, k) = (1, 0)$ and $\langle a_j \rangle_{j=1}^{n} \subseteq \mathbb{N}$. Then for any finite partition of $\mathbb{Z}$, there exist $k \geq m$ and numbers $x, b_1, b_2, \ldots, b_k$ such that

$$\left\{ x, b_1, b_2, \ldots, b_k, \prod_{i=1}^{k} b_i, \left\{ x \cdot \left(a_j^{\sum_{i=1}^{k} b_i} \right) \right\}_{j=1}^{n} \right\}$$

is monochromatic.

Example 3.11. For any $M \in \mathbb{N}$, and for any finite partition of $\mathbb{Z}$, there exist $m, n, p \geq M$, and finite sequences $\langle a_i \rangle_{i=1}^{m}$, $\langle b_j \rangle_{j=1}^{n}$ and $\langle c_q \rangle_{q=1}^{p}$ such that

$$\left\{ \prod_{i=1}^{m} a_i, \prod_{j=1}^{n} b_j, \prod_{q=1}^{p} c_q, \prod_{j=1}^{n} b_j \cdot 2^{\sum_{i=1}^{m} a_i}, \prod_{j=1}^{n} b_j \cdot 3^{\sum_{i=1}^{m} a_i}, \right.$$
$$\left. \prod_{q=1}^{p} c_q \cdot 2^{\left(\sum_{i=1}^{m} a_i\right)\left(\sum_{j=1}^{n} b_j\right)}, \prod_{q=1}^{p} c_q \cdot 3^{\left(\sum_{i=1}^{m} a_i\right)\left(\sum_{j=1}^{n} b_j\right)} \right\}$$

is monochromatic.

4. A new approach to additive and multiplicative operation

In this section, we address patterns involving exponential and symmetric structures. In [24], A. Sisto initiated the study of exponential patterns in Ramsey theory. He proved that for any 2-coloring of $\mathbb{N}$, one of the cells contains the pattern of the form $\{x, y, x^y\}$ and then he conjectured that for any finite coloring, there exists a monochromatic copy of the form $\{x, y, x^y\}$. Finally, in [21], J. Sahasrabudhe proved this conjecture combinatorially, and then in [10], authors found an ultrafilter proof. Subsequent developments have been done in [22,11]. In [13], the authors found several new proof of these results, including a detailed introduction to exponential ultrafilters. Then in [14], some refinements have been made. These works inspire us to introduce two new operations over $\mathbb{N}$. Both these operations are deeply related to exponentiation of symmetric patterns.

4.1. A new additive operation

For $n \in \mathbb{N}$, define the function $f : \mathbb{N} \to \omega$ by $f(n) = \max\{x : 2^x | n\}$. If $n \in \mathbb{N}$, then $n = 2^{x_n}(2y_n - 1)$, where $x_n = f(n)$. It is easy to verify that the function $\varphi : \mathbb{N} \to \omega \times \mathbb{N}$ defined by

$$\varphi(n) = (x_n, y_n) = \left(f(n), \frac{1}{2}\left(\frac{n}{2^{f(n)}} + 1 \right) \right),$$

is a bijection.

Take the commutative semigroup $(\omega \times \mathbb{N}, +)$, where the operation $+$ is defined as $(a,b)+(c,d) = (a+b, c+d)$. Then the bijection φ induces an associative operation $\oplus$ on $\mathbb{N}$, defined by $p = m \oplus n \iff p = 2^{f(m)+f(n)}\left(\frac{m}{2^{f(m)}} + \frac{n}{2^{f(n)}} + 1\right)$. Now define $m \oplus n = 2^{f(m)+f(n)}\left(\frac{m}{2^{f(m)}} + \frac{n}{2^{f(n)}} + 1\right)$.

It can be easily seen that

$$a_1 \oplus a_2 \oplus \cdots \oplus a_n = 2^{\sum_{i=1}^n f(a_i)}\left(\sum_{i=1}^n \frac{a_i}{2^{f(a_i)}} + (n-1)\right).$$

The following theorem addresses monochromatic exponential and symmetric patterns.

Theorem 4.1. *Let $r \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n$ be distinct natural numbers. Then for every r-coloring of $\mathbb{N}$, there exist x, y and c in $\mathbb{N}$ such that*

$$\left\{x 2^{cf(a)}\left(y + c\frac{a}{2^{f(a)}}\right) : a \in \{a_1, a_2, \ldots, a_n\}\right\}$$

is monochromatic.

Proof. Let $\mathbb{A} = \{a_1, a_2, \ldots, a_n\}$ and $r \in \mathbb{N}$. Then choose the Hales-Jewett number $N = N(\mathbb{A}, r)$.

Now consider the word space $\mathbb{A}^N$, and take the correspondence map $g : \mathbb{A}^N \to \mathbb{N}$ defined by

$$g(a_1, a_2, \ldots, a_N) = a_1 \oplus a_2 \oplus \cdots \oplus a_N = 2^{\sum_{i=1}^N f(a_i)}\left(\sum_{i=1}^N \frac{a_i}{2^{f(a_i)}} + (N-1)\right).$$

Now every r-partition on $\mathbb{N}$ induces a r-partition on $\mathbb{A}^N$. Then from Hales-Jewett theorem and above configuration, there exist $c, x, y \in \mathbb{N}$ such that

$$\left\{x 2^{cf(a)}\left(y + c\frac{a}{2^{f(a)}}\right) : a \in \mathbb{A}\right\}$$

is monochromatic. □

The following examples are new.

Example 4.2. For two odd primes p, q, consider the numbers $pq, 2p$ and $2q$. Now $f(pq) = 0, f(2p) = f(2q) = 2$ and $\frac{pq}{2^{f(pq)}} = pq$, $\frac{p}{2^{f(p)}} = p$ and $\frac{q}{2^{f(q)}} = q$.

Then for any r-partition of $\mathbb{N}$, there exist $c, x, y \in \mathbb{N}$ such that

$$\{x(y+cpq), 2^c x(y+cp), 2^c x(y+cq)\}$$

is monochromatic.

Example 4.3. For an odd prime p, consider the numbers $2^3p, p^3$. Now $f\left(p^3\right) = 0, f\left(2^3p\right) = 3$ and $\frac{p^3}{2^{f\left(p^3\right)}} = p^3$, $\frac{2^3p}{2^{f\left(2^3p\right)}} = p$. Then for any r-partition of $\mathbb{N}$, from the above theorem there exist $c, x, y \in \mathbb{N}$ such that

$$\left\{x\left(y + cp^3\right), x2^{3c}\left(y + cp\right)\right\}.$$

is monochromatic. Let $a = xy$, and $b = xc$. Then for any r-partition of $\mathbb{N}$, from the above theorem there exist $c, a, b \in \mathbb{N}$ such that

$$\left\{a8^c + bp, a + bp^3\right\}$$

is monochromatic.

4.2. A new multiplicative operation

Consider the set $\omega \times 2\mathbb{N}$, and the binary operation "$\cdot$" on $\omega \times 2\mathbb{N}$, where the operation "$\cdot$" is defined as component wise multiplication. Then $(\omega \times 2\mathbb{N}, \cdot)$ forms a commutative semigroup. Again each $n \in \mathbb{N}$ can be written as $n = 2^{x_n}\left(2y_n - 1\right)$ in a unique way, where $x_n = f\left(n\right)$. So, the function $\rho : \mathbb{N} \to \omega \times 2\mathbb{N}$ is defined by

$$\rho\left(n\right) = \left(x_n, 2y_n\right) = \left(f\left(n\right), \frac{n}{2^{f(n)}} + 1\right)$$

is a bijection and it induces an associative operation $\otimes$ on $\mathbb{N}$, defined by, $p = m \otimes n \iff p = 2^{f(m)\cdot f(n)} \cdot \mathfrak{G}_{1,1}\left(\frac{m}{2^{f(m)}}, \frac{n}{2^{f(n)}}\right)$. Define $m \otimes n = 2^{f(m)\cdot f(n)} \cdot \mathfrak{G}_{1,1}\left(\frac{m}{2^{f(m)}}, \frac{n}{2^{f(n)}}\right)$.

It can be easily verified that

$$a_1 \otimes a_2 \otimes \cdots \otimes a_n = 2^{\prod_{i=1}^{n} f(a_i)} \cdot \mathfrak{G}_{1,1}\left(\frac{a_1}{2^{f(a_1)}}, \frac{a_2}{2^{f(a_2)}}, \ldots, \frac{a_n}{2^{f(a_n)}}\right).$$

The following theorem is a variant of Theorem 4.1. and addresses new patterns.

Theorem 4.4. *Let $r \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n$ be distinct natural numbers. Then for every r-coloring of $\mathbb{N}$, there exist x, y and c in $\mathbb{N}$ such that*

$$\left\{x2^{f(a)^c}\mathfrak{G}_{1,1}\left(y, \left(\frac{a}{2^{f(a)}}\right)^{(c)}\right) : a \in \{a_1, a_2, \ldots, a_n\}\right\}$$

is monochromatic.

Proof. Let $\mathbb{A} = \{a_1, a_2, \ldots, a_n\}$ and $r \in \mathbb{N}$. Then choose the Hales-Jewett number $N = N\left(\mathbb{A}, r\right)$.

Now consider the word space $\mathbb{A}^N$ and take the correspondence map $g : \mathbb{A}^N \to \mathbb{N}$ defined by,

$$g(a_1, a_2, \ldots, a_N) = a_1 \otimes a_2 \otimes \cdots \otimes a_N = 2^{\prod_{i=1}^{N} f(a_i)} \mathfrak{G}_{1,1}\left(\frac{a_1}{2^{f(a_1)}}, \frac{a_2}{2^{f(a_2)}}, \ldots, \frac{a_N}{2^{f(a_N)}}\right).$$

Now every r-partition on $\mathbb{N}$ induces a r-partition on $\mathbb{A}^N$. Let c be the number of variable positions. Let $d = N - c$, then consider

$$y = \mathfrak{G}_{1,1}\left(\frac{b_1}{2^{f(b_1)}}, \frac{b_2}{2^{f(b_2)}}, \ldots, \frac{b_d}{2^{f(b_d)}}\right)$$

and $x = \prod_{i=1}^{d} f(b_i)$, where $b_i's$ are the in non-variable positions. Then, from the Hales-Jewett Theorem 2.1 and the above expression,

$$\left\{x 2^{f(a)^c} \cdot \mathfrak{G}_{1,1}\left(y, \left(\frac{a}{2^{f(a)}}\right)^{(c)}\right) : a \in \mathbb{A}\right\}$$

is monochromatic. □

The following two examples are immediate consequence of Theorem 4.4.

Example 4.5. Let $n, r \in \mathbb{N}$ and $p_1, p_2, \ldots, p_n$ be different odd primes. Then for any r-coloring of $\mathbb{N}$, there exist $x, z, c \in \mathbb{N}$ such that

$$\{xz \cdot (p_i + 1)^c - x : i \in \{1, 2, \ldots, n\}\}$$

is monochromatic.

Example 4.6. Let $n, r \in \mathbb{N}$ and p be an odd prime. Then for any r-coloring of $\mathbb{N}$, there exist $x, z, c \in \mathbb{N}$ such that

$$\left\{xz \cdot \left(p^i + 1\right)^c - x : i \in \{1, 2, \ldots, n\}\right\}$$

is monochromatic

Declaration of competing interest

We don't have any competing interest with anyone.

Data availability

No data was used for the research described in the article.

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